ON RAYLEIGH-TAYLOR Z-PINCH INSTABILITY

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It is well known that in the initial stage of a powerful pulse discharge in a gas the discharge boundary moves with acceleration towards the axis. In such a case Rayleigh-Taylor instability [1] may play a significant part. In some experimental papers on the Z-pinch, even before the moment of first constriction in fact, an instability of the surface discharge with respect to pinch-type perturbations has been detected with an instability increment which turns out to be approximately equal to half the quantity

$$\omega_0 = (gk)^{1/2}. \qquad (1)$$

Here g is the acceleration of the discharge boundary, and k is the wave number of the perturbation harmonic. Expression (1) was obtained in the well-known paper of Kruskal and Schwarzschild [5], in which Rayleigh-Taylor instability of a semi-infinite plasma contained by a magnetic field in a uniform gravitational field was studied in particular. In deriving (1) it was assumed that the plasma lies above the horizontal plane $y = 0$, and that the accelerating gravity force, the magnetic field and the wave vector of the surface disturbance of the plasma have only the components $g_y = -g$, B_z, and $k_x = k$, respectively.

It is of definite interest to find a direct solution to the problem of the Rayleigh-Taylor Z-pinch instability in the constriction process. The present paper solves this problem for one of the possible Z-pinch models in the initial stage and, in particular, for the "snowplough" model [6].

We note that Z-pinch instability up to the moment of first constriction has been considered in [7] for various models, including the "snow-plough." However, in this paper only radial perturbations of the plasma surface were allowed. Rayleigh-Taylor instability did not appear in the calculations of [7] because of this restriction, although other types of instability were investigated in the proper manner.

We shall therefore consider an infinite circular cylinder composed of a perfectly conducting plasma at zero pressure. A current creating an azimuthal magnetic field flows over the surface of the cylinder along the z axis. The current surface is dragged towards the axis under the influence of the Lorentz force, entraining the plasma and gathering it into an infinitely thin surface layer. We shall consider that the discharge boundary moves towards the axis with a constant acceleration g. This assumption has been experimentally confirmed for the initial stage of constriction [4]. The radius of the current shell $R_0(t)$ is associated with the initial radius of the cylinder $R_* = R_0(0)$ by the kinematic relation

$$R_0(t) = R_* - \frac{1}{2}gt^2 = R_*\left(1 - \frac{t^2}{t_*^2}\right) \qquad \left(t_* = \left(\frac{2R_*}{g}\right)^{1/2}\right). \qquad (2)$$

Here t_* is the time in which the discharge boundary moving with a constant acceleration g should have reached the axis.

The equation of motion of an element of surface area $dA_0 = R_0 d\varphi dz$ has the form

$$\frac{d}{dt}\left(\frac{M_0}{2\pi R_0} R_0 d\varphi\, dz \frac{dR_0}{dt}\right) = -p_0 R_0 d\varphi\, dz \qquad \left(p_0(t) = \frac{B_0^2(t)}{8\pi}\right). \qquad (3)$$

Here $p_0(t)$ is the magnetic pressure on the surface of the plasma, and $M_0(t)$ is the mass per unit length of the cylinder gathered up by the boundary. Clearly,

$$M_0(t) = \pi\rho(R_*^2 - R_0^2), \qquad (4)$$

where ρ is the initial plasma density. Equation (3) gives

$$p_0 = \frac{3}{2}\rho g^2 t^2 \frac{1 - 5/6\,(t/t_*)^2}{1 - (t/t_*)^2}, \qquad (5)$$

when (2) and (4) are taken into account.

In accordance with the model we have chosen we shall consider further only the initial stage of constriction when $t \ll t_*$; accordingly, we shall everywhere neglect quantities of the order of $(t/t_*)^2$ in comparison with unity. Then (5) becomes

$$p_0 = \frac{B_0^2}{8\pi} = \frac{3}{2}\rho g^2 t^2.$$

Whence

$$B_0 = 2(3\pi\rho)^{1/2} gt,$$

and consequently the total discharge current should be proportional to time

$$I_0(t) = {}^1/_2\, cB_0 R_0 = (3\pi\rho)^{1/2} cgR_* t.$$

Such an increase of current does, in fact, occur in the initial stage of a discharge.

We shall now investigate the stability of the discharge shell. Let $\xi(\varphi, z, t) = (\xi_r, \xi_\varphi, \xi_z)$ be the small displacement of a particle initially situated on the surface of the cylinder at the point (φ, z); $\mathbf{n} = \mathbf{e}_r + \mathbf{n}_1$ a unit vector normal to the surface; and dA an element of area of the perturbed surface of the cylinder. Then in an approximation which is linear with respect to the perturbation

$$\mathbf{n}dA = [\mathbf{e}_\varphi(R_0 + \xi_r)d\varphi + \xi(\varphi + d\varphi, z) -$$

$$-\xi(\varphi, z)] \times [\mathbf{e}_z dz + \xi(\varphi, z + dz) - \xi(\varphi, z)] = \qquad (6)$$

$$= R_0 d\varphi\, dz\left[\mathbf{e}_r\left(1 + \frac{1}{R_0}\xi_r + \frac{1}{R_0}\frac{\partial\xi_\varphi}{\partial\varphi} + \frac{\partial\xi_z}{\partial z}\right) - \mathbf{e}_\varphi \frac{1}{R_0}\frac{\partial\xi_r}{\partial\varphi} - \mathbf{e}_z\frac{\partial\xi_r}{\partial z}\right]$$

Whence

$$dA = R_0 d\varphi\, dz\left(1 + \frac{1}{R_0}\xi_r + \frac{1}{R_0}\frac{\partial\xi_\varphi}{\partial\varphi} + \frac{\partial\xi_z}{\partial z}\right),$$

$$\mathbf{n}_1 = \left(0, \quad -\frac{1}{R_0}\frac{\partial\xi_r}{\partial\varphi}, \quad -\frac{\partial\xi_r}{\partial z}\right)$$

We find the perturbation of the magnetic field $\mathbf{B}_1$ outside the plasma from the equations

$$\mathbf{B}_1 = \nabla\psi_1, \qquad \Delta\psi_1 = 0 \qquad (7)$$

and from the boundary condition for the total field $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ on the surface of the plasma

$$0 = \mathbf{nB} \approx \mathbf{e}_r\mathbf{B}_1 + \mathbf{n}_1\mathbf{B}_0. \qquad (8)$$

We shall consider that all quantities depend on φ and c in the following manner

$$e^{i(m\varphi + kz)}. \qquad (9)$$

Then the solution of the equation $\Delta\psi_1 = 0$, vanishing at infinity and satisfying condition (8), is

$$\psi_1 = \frac{imB_0 K_m(kr)}{kR_0 K_m'(kR_0)}\xi_r, \qquad (10)$$

where $K_m(kr)$ is Macdonald's function, and the prime indicates differentiation with respect to the argument. For the pressure of the magnetic field on the perturbed surface of the cylinder we have

$$8\pi p = \mathbf{B}^2(R_0 + \xi_r) \approx \mathbf{B}_0^2(R_0) + 2\mathbf{B}_0(R_0)\mathbf{B}_1(R_0) + \xi_r\frac{\partial}{\partial r}\mathbf{B}_0^2(R_0),$$

whence, using (7) and (10), we find

$$p = p_0 + p_1 = \frac{B_0^2}{8\pi} - \frac{B_0^2}{4\pi}\left[1 + \frac{m^2 K_m(kR_0)}{kR_0 K_m'(kR_0)}\right]\frac{\xi_r}{R_0}, \qquad (11)$$

where the second term obviously represents the pressure perturbation.

The equation of motion of the surface element having mass

$(M_0/2\pi)\,d\varphi dz$ and area* dA after perturbation has the form

$$\frac{d}{dt}\left[\frac{M_0}{2\pi}\,d\varphi\,dz\left(\frac{dR_0}{dt}\mathbf{e}_r+\frac{\partial\xi}{\partial t}\right)\right]=-(p_0+p_1)\,\mathbf{n}\,dA.$$

Whence, taking into account the equation for the unperturbed motion (3), and also equations (4), (6) and (11), we obtain in an approximation which is linear with respect to the perturbation

$$\frac{1}{2}\rho\frac{d}{dt}\left[(R_*^2-R_0^2)\frac{\partial\xi}{\partial t}\right]= \tag{12}$$

$$=p_0\left\{\left[\left(1+\frac{2m^2K_m(kR_0)}{kR_0K_m'(kR_0)}\right)\xi_r-\frac{\partial\xi_\varphi}{\partial\varphi}-R_0\frac{\partial\xi_z}{\partial z}\right]\mathbf{e}_r+\frac{\partial\xi_r}{\partial\varphi}\mathbf{e}_\varphi+R_0\frac{\partial\xi_r}{\partial z}\mathbf{e}_z\right\}.$$

Further, if we make use of Eq. (2) and the dependence of all quantities on φ and z in the form given in (9), then for the new variable

$$\eta = t\xi \tag{13}$$

we obtain from (12) the system of equations

$$\eta_r^{\cdot\cdot}-\frac{3g}{R_*}\left[\left(1+\frac{2m^2K_m(kR_*)}{kR_*K'_m(kR_*)}\right)\eta_r-im\eta_\varphi-ikR_*\eta_z\right]=0,$$

$$\eta_\varphi^{\cdot\cdot}-\frac{3g}{R_*}im\eta_r=0,\qquad \eta_z^{\cdot\cdot}-3gik\eta_r=0. \tag{14}$$

In accordance with the remark made following Eq. (5), R_* replaces R_0 everywhere. Taking (13) and the initial condition $\xi(0)=\xi_0$, into account, we seek $\eta(t)$ in the form

$$\eta(t)=\xi_0\frac{\operatorname{sh}\omega t}{\omega}. \tag{15}$$

Setting (15) in (14) gives the system of algebraic equations

$$\left[\omega^2-\frac{3g}{R_*}\left(1+\frac{2m^2K_m}{kR_*K_m'}\right)\right]\xi_{0r}+\frac{3g}{R_*}im\xi_{0\varphi}+3gik\xi_{0z}=0,$$

$$\omega^2\xi_{0\varphi}-\frac{3g}{R_*}im\xi_{0r}=0,\qquad \omega^2\xi_{0z}-3gik\xi_{0r}=0. \tag{16}$$

A nontrivial solution of (16) exists if

$$\omega^4-\frac{3g}{R_*}\left(1+\frac{2m^2K_m}{kR_*K_m'}\right)\omega^2-\left(\frac{3gm}{R_*}\right)^2-(3gk)^2=0,$$

whence

$$\omega^2=3g\left\{\frac{1}{2R_*}\left(1+\frac{2m^2K_m}{kR_*K_m'}\right)\pm\left[\frac{1}{4R_*^2}\left(1+\frac{2m^2K_m}{kR_*K_m'}\right)^2+\frac{m^2}{R_*^2}+k^2\right]^{1/2}\right\}.$$

Thus Eq. (12) has a solution of the form

$$\xi=\xi_0\frac{\operatorname{sh}\omega t}{\omega t} \tag{17}$$

containing the unstable mode with

$$\omega^2=3g\left\{\frac{1}{2R_*}\left(1+\frac{2m^2K_m}{kR_*K_m'}\right)+\right.$$

$$\left.+\left[\frac{1}{4R_*^2}\left(1+\frac{2m^2K_m}{kR_*K_m'}\right)^2+\frac{m^2}{R_*^2}+k^2\right]^{1/2}\right\}. \tag{18}$$

Before examining the relationships obtained, we note that by setting M_0 = const we arrive at the problem solved in [8] on the instability of a radially accelerated thin cylindrical plasma shell of constant mass. In this case the unstable solution of Eq. (12) has the form $\xi=\xi_0e^{\nu t}$, where ν is related to c(ω) from (18) by $3\nu^2=\omega^2$.

The increase with time of the mass of the moving Z-pinch shell exerts a substantial influence on the character of the instability. It follows from (18) that the unstable perturbation increases slowly for $\omega t<1$ and increases almost exponentially with an increment close to ω for $\omega t\gg 1$. In order to judge whether the perturbation which arises for t = 0 will develop during the time under consideration $t\ll t_*$, we introduce the instability "increment" ω_e, a quantity which is the reciprocal of the time during which the initial perturbation increases by a factor of e. For (17) this increment is $\omega_e=0.37\,\omega$. Keeping this in mind, we shall make a further direct examination of the quantity ω given by Eq. (18).

We shall consider some particular cases. First, k = 0, $m\neq 0$. Since

$$\lim_{k\to 0}\frac{K_m(kR_*)}{kR_*K_m'(kR_*)}=-\frac{1}{m},$$

Eq. (18) becomes

$$\omega^2=\frac{3g}{2R_*}\{1-2m+[(1-2m)^2+4m^2]^{1/2}\}. \tag{19}$$

It follows from this that the surface of the discharge for k = 0 is unstable for all values of m, which is quite unexpected. It is true that for $m\sim 1$ we have from Eq. (19)

$$\omega^2\sim\frac{g}{R_*}\sim\frac{1}{t_*^2},$$

i.e., the perturbations grow during a time of the order of t_* which lies outside the scope of our treatment. However, if $m\gg 1$, then

$$\omega^2\approx\frac{3gm}{R_*}(\sqrt{2}-1)\sim\frac{m}{t_*^2}\gg\frac{1}{t_*^2},$$

so that shortwave perturbations increase over times which are small compared with ct_*. Moreover, disturbances with $m\neq 0$ bend the lines of force increasing the energy of the magnetic field, so that at least such perturbations as are of sufficiently short wavelength should be unstable. The contradiction which results is associated with the neglect of the thickness of the moving discharge shell. As shown in [8], taking into account the finite thickness of the shell leads to the appearance of short-wave perturbation instability. If we apply the result of [8] to our model, then for k = 0 only perturbations with $m<R_*/2a$, will be unstable, where a is the thickness of the accelerated sheet. Because of the inadequacy of the model we have chosen, in the general case, when $k\neq 0$ and $m\neq 0$, Eq. (18) displays instability for all values of k and m.

We shall further consider pinch-type instabilities for which m = 0, $k\neq 0$. From the point of view of energy these pinches are the most dangerous, since they bend the lines of force of the magnetic field. It follows from Eq. (18) that in this case

$$\omega^2=3g\left[\frac{1}{2R_*}+\left(\frac{1}{4R_*^2}+k^2\right)^{1/2}\right]. \tag{20}$$

We shall determine the part played by the various terms in the expression obtained. If we allow only radial displacement of the particles in the surface of the plasma, i.e., $\boldsymbol{\xi}=(\xi_r,0,0)$, then instead of system (14) there will be only one equation

$$\eta_r^{\cdot\cdot}-\frac{3g}{R_*}\eta_r=0$$

and correspondingly instead of (20)

$$\omega_r^2=\frac{3g}{R_*}=\frac{6}{t_*^2}.$$

Thus the R_*-terms are associated with the instability of purely radial perturbations and their presence in (20) expresses the well-known fact that the magnetic pressure on the surface of a body of revolution with a longitudinal current is greater where the radius is less. Since $\omega_r\sim 1/t_*$, in the initial stage of discharge treated ere an instability of this type is of no significance.

On the other hand, for $R_*\to\infty$, on passing to a plasma with a flat boundary, (20) becomes

$$\omega^2=3gk. \tag{21}$$

Using expression (1) for the increment of the Rayleigh-Taylor instability of a semi-infinite plasma in a gravitational field, we write

*Taking into account the change of mass on perturbation complicates the equations considerably.

(21) in the form

$$\omega^2 = 3\omega_0^2. \tag{22}$$

Hence we may conclude that the term with k in (20), due to the displacements of particles along the surface of the plasma, corresponds to the Rayleigh-Taylor instability. For sufficiently short-wave perturbations, when $kR_* \gg 1$, we have (21) and (22) instead of (20) and the "increment" we have introduced $\omega e \approx 0.64\omega_0$.

Here the time for development of the instability is of the order of

$$\frac{1}{\omega} = \frac{t_*}{(6kR_*)^{1/2}} \ll t_*.$$

Consequently, in the "snow-plough" model the Rayleigh-Taylor Z-pinch instability may play a significant part at the constriction stage.

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